The Definition of the Cartesian Product Is Weird

Let A and B be sets. Then the Cartesian product $A \times B$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

This definition is very basic and easy to understand. It is also familiar even to people who haven't studied mathematics to the level where things start having definitions instead of descriptions: the Cartesian product describes what is going on when you graph things on the Cartesian plane.

But I find the definition of the Cartesian product weird because it doesn't actually tell you what the Cartesian product is. It doesn't, as far as I can tell, actually *define* the Cartesian product. Instead, it uses suggestive notation to influence the reader in such a way that they are *unlikely* to *mess up* what the Cartesian product is.

Here is the basic issue with the definition: the above definition of the Cartesian product tells you *what elements* are in the set $A \times B$, and nothing else. Because of the suggestive notation (a, b) given to each element, knowing what elements are in the set feels meaningful, as each element clearly appears to pertain to the sets A and B. So if $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$, then an element such as $(a_1, b_2) \in A \times B$ feels meaningful because we can see how its components a_1 and b_2 relate back to A and B respectively.

But logically, there's no "meaning" to elements. They're just squiggles that are put inside braces and separated by commas from other squiggles in the same pair of braces. It feels more sensible to say "For all x in X" than "For all y in X", but mathematically it's the same.

There's nothing wrong with suggestive notation, of course, but it means that a definition that relies on *how elements are written* to define the set is a definition that is being sneaky. If the Cartesian product $A \times B = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)\}$, then we could have just as easily written

$A \times B = \{w, x, y, z\}$ because in each case, we've filled a pair of braces with four squiggles separated by commas.

Of course, the elements w, x, y, z don't have anything to do with the sets A and B, but that's also true of the elements $(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)$. The latter elements *appear* to be related to A and B, but the element (a_1, b_1) isn't *actually* a complex object made of two parts, one part coming from A and the other part coming from B. It's just an *arbitrary squiggle* that *happens* to *visually resemble* the kind of thing we want to be talking about when we talk about the Cartesian product of A and B. But if we accept the obvious-sounding principle that no theorem depends on a choice of squiggle—the principle that {*one, two, three*} could just as easily be {*shmlep, bloop, blep*}—then we should be able to talk about the Cartesian product without relying on suggestive notation to fill in the gap left by this so-called definition.

The definition of the Cartesian product does do one important thing: it tells you how many elements are in the Cartesian product. It obscures why this matters and how you can mess around with it, but it is important to have a way of pinning down how many elements the Cartesian product should have, even if you aren't told why it should have that many, or what "should" means in this context.

As best I can tell, the real definition of the Cartesian product is "any set that labels the points on the Cartesian plane":

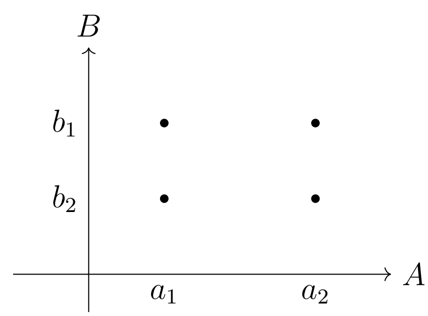

Figure 1

You could think of the points on this graph as constituting a "generic" definition of the Cartesian product of A and B, where anything that labels these four points is a specific Cartesian product. So the set $\{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)\}$ works as a Cartesian product because we can label the elements like so:

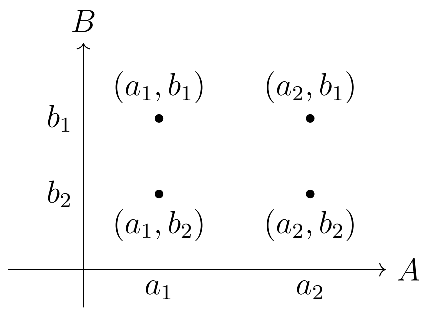

Figure 2

But we can clearly also use the set $\{w, x, y, z\}$ to label the points on the graph:

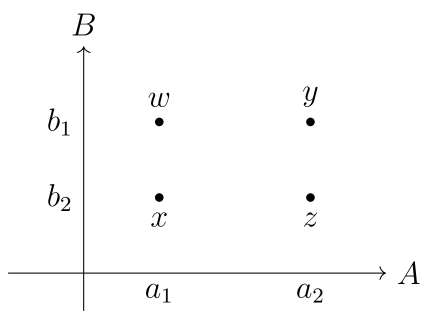

Figure 3

Clearly, any four-element set is going to work as the Cartesian product of A and B. Note also that there are many ways to use a four-element set to label the points on the graph. (Specifically, there are $4! = 24$ ways to do so for a given four-element set.) For example, both of the following graphs are also the Cartesian product of A and B:

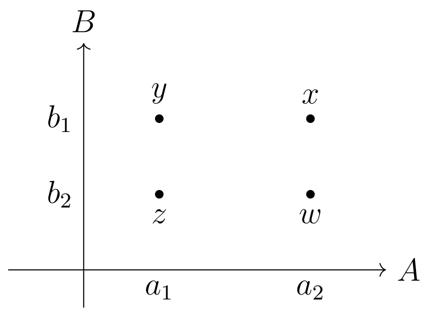

Figure 4

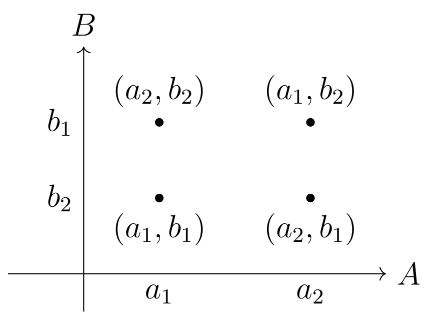

Figure 5

The second graph might be confusing because we label the points "wrong"—for example, the point where a straight line from a_1 intersects with a straight line from b_2 is labeled (a_1, b_1). But remember, the label is an element of a set, and an element of a set is just a squiggle. So we can label it however we like. It's preferable for humans to use notation that "makes sense", but in purely mathematical terms, it makes no difference.

This gets even weirder, however, when we see that *any* set can serve as the Cartesian product. Let $C = \{a, b, c, d, e, f, g, h\}$. Then we can use C to fill in the graph like so:

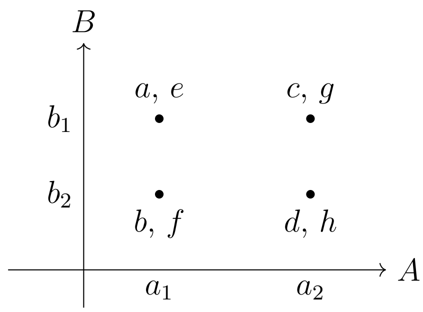

Figure 6

Here, we have labeled every point *twice* rather than once like we did previously. Is this...a problem? Are we not allowed to do this? The definition of the Cartesian product doesn't say.

There's nothing *illogical* about this double-labeling—it doesn't break any mathematical rules—but it isn't *efficient*. It's clearly much more sensible to label each point exactly once. So this eight-element set can "do the job" of the Cartesian product in terms of labeling every point on the graph, but it isn't "the" Cartesian product because it can't label every point on the graph *efficiently*. Only a four-element set can be an efficient Cartesian product (of two sets with two elements each).

We can see that the definition of the Cartesian product gives us a way of constructing a set with exactly the right amount of elements. In fact, this is the *only* thing that the definition of the Cartesian product does that's meaningful on a purely mathematical level. This makes sense because any set in isolation can only be described in terms of its cardinality, so if you're defining a set, all you can really do is define its cardinality. But giving you a way to construct a set with the right amount of elements isn't enough to define the Cartesian product because "any four-element set" isn't quite sufficient to label all four points on the graph. We have to actually *use* the set to label all four points on the graph. Otherwise we might end up with the following situation:

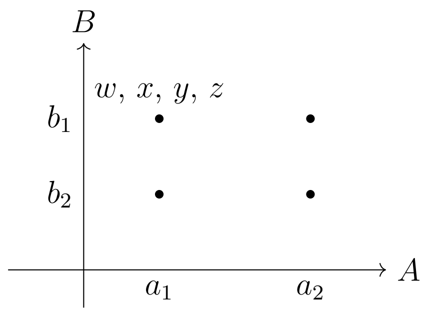

Figure 7

We can clarify what we mean by "using" the set in this way by talking about projection functions. Each label's position on the graph indicates two mapping from the four-element set: one mapping projecting the set onto A by drawing a vertical line straight down, and one mapping projecting the set onto B by drawing a horizontal line to the left. Wherever you land on the A-axis is where the element is mapped to A, and wherever you land on the B-axis is where the element is mapped to A. For example, Figure 7 depicts a pair of functions $f : \{w, x, y, z\} \to A$ and $g : \{w, x, y, z\} \to B$ where

$$
\begin{gathered}
w \overset{f}{\mapsto} a_1 \\
w \overset{g}{\mapsto} b_1 \\
x \overset{f}{\mapsto} a_1 \\
x \overset{g}{\mapsto} b_1 \\
y \overset{f}{\mapsto} a_1 \\
y \overset{g}{\mapsto} b_1 \\
z \overset{f}{\mapsto} a_1 \\
z \overset{g}{\mapsto} b_1.
\end{gathered}
$$

Whereas Figure 2 depicts a pair of functions $p_1 : A \times B \to A$ and $p_2 : A \times B \to B$ where

$$
\begin{aligned}
(a_1, b_1) &\overset{p_1}{\mapsto} a_1 \\
(a_1, b_1) &\overset{p_2}{\mapsto} b_1 \\
(a_2, b_1) &\overset{p_1}{\mapsto} a_2 \\
(a_2, b_1) &\overset{p_2}{\mapsto} b_1 \\
(a_1, b_2) &\overset{p_1}{\mapsto} a_1 \\
(a_1, b_2) &\overset{p_2}{\mapsto} b_2 \\
(a_2, b_2) &\overset{p_1}{\mapsto} a_2 \\
(a_2, b_2) &\overset{p_2}{\mapsto} b_2.
\end{aligned}
\tag{1}
$$

Clearly, projection functions like p_1 and p_2 are crucial for having a four-element set like $A \times B$ actually serve as the Cartesian product of A and B. But the definition of the cartesian product where $A \times B$ is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ *does not mention projection functions at all!* Instead, it *hints* at projection functions by using suggestive notation such that the reader can be expected to infer, probably unwittingly, what the projection functions are supposed to be.

In economics, by the way, you will hear people say that math is useful because it makes you clearly state your assumptions.

The suggestiveness of the notation in the set $A \times B$ lies in the fact that the elements of $A \times B$ have components that are written the same way as the elements of A and B are, in a way that suggests obvious projection functions. For example, the ordered pair $(a_1, b_1) \in A \times B$ obviously feels like it should be mapped to a_1 with one projection function and be mapped to b_1 with another projection function. Projection functions that perform the "obvious" mapping—so obvious it doesn't even need to be described—from the Cartesian product to the sets it is a product of are called *canonical* projection functions. Because these canonical projection functions are so intuitively obvious, the standard definition of the Cartesian product is able to get away with not mentioning them even though without them, you don't have a Cartesian product, you just have a four-element set.

So the definition of the Cartesian product does exactly two things: It provides you with a way of constructing the relevant set such that it has exactly the right amount of elements, and it gives you a method for writing the elements of the set such that the canonical projection functions that you need to turn the set from just a set into the actual Cartesian product are obvious enough that you will implicitly define them without realizing you've done so.

This works as a definition in the sense that it probably will cause you to do everything right in practice. But it isn't a definition in the sense of actually *defining* the Cartesian product! The Cartesian product of A and B isn't a four-element set, it's a four-element set *that labels each point on the Cartesian plane where* A *is the* x*-axis and* B *is the* y*-axis exactly once.*

But this definition relies on being able to visualize the Cartesian product, which is of limited utility and won't generalize to higher dimensions. So a way of defining the Cartesian product that *actually* defines it is this: For any two sets A and B, the Cartesian product is the set $A \times B$ along with projection functions $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ such that for any other set X with functions $f : X \to A$ and $g : X \to B$, there is a *unique* function $k : X \to A \times B$ such that the following diagram commutes:

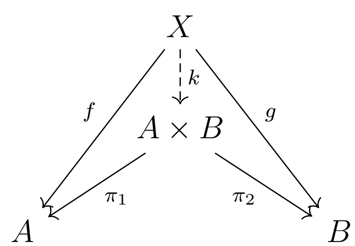

Figure 8

The diagram commutes when $\pi_1 \circ k = f$ and $\pi_2 \circ k = g$.

This definition is much more abstract and hard to understand than the familiar definition of the Cartesian product based on ordered pairs. But the advantage is that it actually *defines* the Cartesian product: you can check your candidate product against this definition to see if it is or is not the Cartesian product.

It's noteworthy that this definition makes explicit mention of projection functions. The ordered pairs definition leaves the projection functions implied. But as you can see from comparing Figure 3 with Figure 7, a set is only a Cartesian product if it is equipped with appropriate projection functions.

You know whether the projection functions are appropriate depending on whether they fit the definition.

For example, let $X = \{x, y, z\}$ with projection functions to A and to B as indicated by this graph:

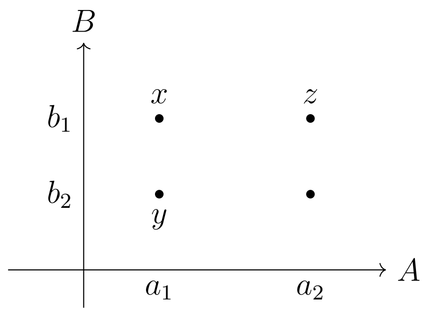

Figure 9

And equip the set $A \times B$ with projection functions like so:

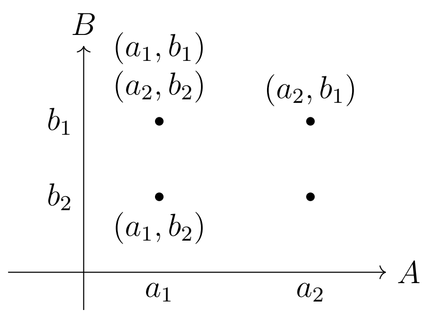

Figure 10

Visually, we can see that we've equipped the set $A \times B$ with the wrong projection functions for $A \times B$ to be the product of A and B. But we can show this formally by recalling that we require a *unique* function $k : X \to A \times B$ such that the way X projects is preserved. This means that each element of X should be mapped onto an element of $A \times B$ that is on the same point in the Cartesian plane. Thus, we require $k(z) = (a_2, b_1$ and $k(y) = (a_1, b_2)$. These choices are forced. But what about $k(x)$? We could map x to (a_1, b_1) or to (a_2, b_2). Because we have two choices for $k(x)$, this means there are two functions from X to $A \times B$ such that the way X projects to A and to B

is preserved. Since there is no unique function that does this preservation, this means that $A \times B$ along with the above projection functions is not the Cartesian product. However, if our candidate product is $A \times B$ along with these projection functions, it will work:

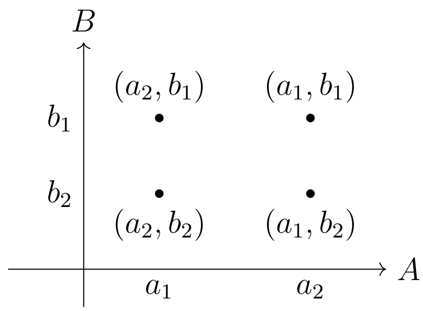

Figure 11

You can see that we must have $k(x) = (a_2, b_1)$, $k(y) = (a_2, b_2)$, and $k(z) = (a_1, b_1)$. Thus, there is a unique function from X to $A \times B$ such that the way X projects to A and to B is preserved. You can verify that any set with a pair of functions to A and to B has only one possible function to $A \times B$ along with the projection functions indicated in Figure 11 such that the diagram in Figure 8 commutes. For example, the set and projection functions indicated in Figure 7 has only one way of mapping to the situation depicted in Figure 11 such that everything commutes.

We can see why the Cartesian product needs to have enough elements—four, when it's the product of two two-element sets. For example, the candidate product depicted in Figure 9 won't be able to make diagrams commute when it needs to receive an element from another set that projects to $a_2 \in A$ and to $b_2 \in B$. What if it has more than four, like the set $W = \{v, w, x, y, z\}$ along with projection functions depicted here:

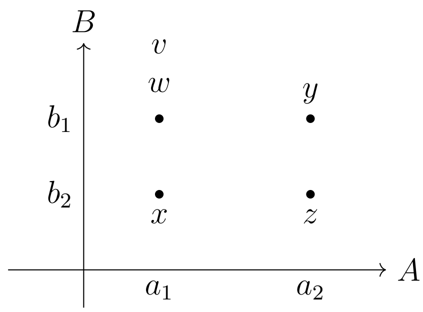

Figure 12

Then you can see that any set with an element that projects to $a_1 \in A$ and to $b_1 \in B$ will have two choices of how to map to W to make everything commute, so we won't have a unique function, so W cannot be a Cartesian product of A and B. Therefore, a Cartesian product of A and B must have exactly four elements. And indeed, the original definition we saw of the Cartesian product did tell us to construct a four-element set to be the product of A and B. But as we've seen, having four elements is merely necessary, not sufficient. You also need the projection functions, which the definition of the Cartesian product only hinted at, but never described explicitly.

And I just think it's interesting that mathematical definitions sometimes aren't really definitions, but in fact leave a crucial half of the definition unstated.